ON OPTIMAL CONTROL OF DYNAMIC SYSTEMS USING THE ALGORITHM OF NUMERICAL METHODS

HRUBINA Kamil – JADLOVSKÁ Anna, SR

Abstract

The paper solves the problem of the dynamic system optimal control or process. Behaviour of the dynamic system is described by the system of differential equations. In order to solve the defined task of optimal control, the author presents the created algorithm based on the successive approximations method. The algorithm was realised within MS Excel environment.

Key words: optimal control, algorithm of the successive approximations method, Hamilton function

1 Introduction

Regarding the development of computer and information technology, requirements for creating qualitatively new control systems, especially that of optimal due to the selected quality index eventually more indexes have been extended. That is why there has been assumed the development of the theory of processes or systems optimal control. Original works of Athans, Bellman, Butkovskij, Lions, Mesarovič, Pontrjagin etc, are considered to be significant in this field of science. [1, 2, 3, 7, 8].
The following authors in their monographs and papers deal with the theory and creation of the algorithm for numerical simulation of mathematical models of dynamic processes. [3, 4, 5, 6].
The aim of the paper is to present theoretical suppositions for creation and realisation of the algorithm using the successive approximations method to the numerical solution of the mathematical model of the dynamic process.

2 Mathematical formulation of the problem

In this paper we are going to consider the dynamic system control. For the part of reality that we are going to control it is necessary to create the adequate model of the object. Thus, we define the system on the object. We create the dynamic system of the model in order to predict the behaviour of the object in the future. In fact, there exist two methods how to create the system on the given object: analytical and experimental. With the analytical method we use different physical, economical and other principles based on which we are searching for the relationships between variables.
We define this method of model creation as mathematical-physical analysis. The other method of system creation follows from measurements made on real object as well as from analysis of measured data in order to determine the relationships between the variables. We call this method of model creation of the investigated object experimental identification. With this method of model creation we have to consider the fact that measurement has always limited accuracy, and consequently, model created by this method is often a stochastic system.
Because the model of the object or process is a system and we are observing time continuance of its variables, we say that we are investigating the dynamic system.

The structure (*S, R*) will be called a system, where S is a set of elements and R is a set of relations (relationships) between them. Taking into consideration Mesarovič's definition of a system, let us introduce the equivalent definition: system is a propositional formula *S(x)* producing true statement $V_1(x)$, $V_2(x)$, ..., $V_m(x)$ on mutual relationships of elements and system environment including state of the system.
Presented definition of the system is of a great practical importance especially for our considerations because in many systems only formal statements or functions, differential equations, integral equations etc are investigated.
We distinguish dynamic systems and static systems according to the state of the system: whether it changes or does not change in time. Dynamic system, state of which can be influenced, is called controllable. If it does not satisfy the mentioned conditions, it is called non-controllable. We are going to deal with the dynamic systems and controllable systems.
With the mathematical models, we cannot work directly with the „state" of the system but only with the certain information on its state. That is why we going to suppose that the state of the system can be described in any instant by *n*-number of ordered real numbers which we are going to call the state vector: $\boldsymbol{x}^T(t) = (x_1(t), x_2(t), ..., x_n(t))$

The set of all feasible states determined by the conditions of certain task will be called state space.
Controllability of dynamic system can be described as follows: for each concrete function $\boldsymbol{u}(t)$ from the set $\boldsymbol{U} \subset \boldsymbol{R}_m$ it is possible according to given dependence definitely to assign the state vector $\boldsymbol{x}(t+\Delta t)$, [4, 6].

Theory of automated control investigates methods by means of which we can influence the system in order the controlled system behaves according to our requirements. Requirements for control can be as follows:

1. Compensation of failure variables influence
2. Problem of regulators (proposition of control structure to stabilize the system)
3. Problem of monitoring (minimisation of regulating deviation)
4. Optimal control (in the modern theory of control the requirements towards control are compiled in quality control criterion and the problem of control is transformed into optimisation problem of minimisation of quality control criterion)

Let the controlled process or system be described by the set of differential equations with the initial conditions and constraints of the following form:

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \qquad (1)$$

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0,\ \boldsymbol{u}(t) \in U,\ t \geq t_0$$

where $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ is n-dimensional vector of the phase coordinates, $\boldsymbol{u} = (u_1, \ldots, u_m)$, m-dimensional vector of control functions, t is time, $\boldsymbol{f} = (f_1, \ldots, f_n)$ is a given vector function, $\boldsymbol{x}_0$ – is a constraint vector, t_0 – the initial time, U is a closed set of m-dimensional space. Partially continuous functions $\boldsymbol{u}(t)$ which satisfy the constraints (1) will be called the feasible controls.
The task is to determine the feasible control $\boldsymbol{u}(t)$ which minimizes the following form of the functional:

$$J(\boldsymbol{u}) = (\boldsymbol{c}, \boldsymbol{x}(t_1)),\ t_1 > t_0 \qquad (2)$$

where t_1 is a given moment of time, $\boldsymbol{c} = (c_1, \ldots, c_n)$ is non-zero constant vector; the dot product is introduced by brackets (,). We will assume that the defined problem has its solution in the domain of feasible controls $\boldsymbol{u}(t)$ and we will call them the optimal control.
Let us define n-dimensional vector $\boldsymbol{p} = (p_1, \ldots, p_n)$ of the adjoint variables and the Hamilton function H. Let us introduce the adjoint system and the conditions of transversality:

$$H(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t) = (\boldsymbol{p}(t),\ \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \qquad (3)$$

$$\frac{d\,p_i}{d\,t} = \frac{\partial H}{\partial x_i} = -\sum_{j=1}^{n} p_j \frac{\partial f_j(\boldsymbol{x}, \boldsymbol{u}, t)}{\partial x_i};$$

$$\boldsymbol{p}\,(t_1) = -\,\boldsymbol{c} \qquad (4)$$

According to the maximum principle the searched optimal control minimizes the function H in the relation (3) for $\boldsymbol{u} \in U$ when $t \in \langle t_0, t_1\rangle$ is arbitrary, if $\boldsymbol{x}$, $\boldsymbol{p}$ satisfy the equations and constraints (1), (4).

3 The methods of solution and algorithms

For the defined problem optimal control solution let us introduce at first the simpler variant of the successive approximations method [1]:

The k-th iteration contains these steps ($k = 1, 2, \ldots$):

1. using control $\boldsymbol{u} = \boldsymbol{u}^{(k)}(t)$ we can solve the Cauchy problem (1) in order to define the trajectory $\boldsymbol{x} = \boldsymbol{x}^{(k)}(t)$ at the interval $\langle t_0, t_1\rangle$;
2. we solve the Cauchy problem in “reverse time” from the time $t = t_1$ to $t = t_0$ when $\boldsymbol{u} = \boldsymbol{u}^{(k)}(t)$, $\boldsymbol{x} = \boldsymbol{x}^{(k)}(t)$ and we will define the adjoint variables $\boldsymbol{p}^{(k)}(t)$ at the interval $\langle t_0, t_1\rangle$;
3. we will define the control $\boldsymbol{u}^{(k+1)}(t)$ at the interval $\langle t_0, t_1\rangle$ from the condition

$$H(\boldsymbol{x}^{(k)}(t), \boldsymbol{u}^{(k+1)}(t), \boldsymbol{p}^{(k)}(t), t) = \max H(\boldsymbol{x}^{(k)}(t), \boldsymbol{u}^{(k)}(t), \boldsymbol{p}^{(k)}(t), t) \qquad (5)$$

If the condition (5) defines $\boldsymbol{u}^{(k+1)}(t)$ by multivalent way, we will chose the arbitrary possible value and we will go to the following iteration according to the algorithm of successive approximations:

Step 1. Reading of the output values

Step 2. Solution of the set of differential equations

Step 3. Calculation of the optimal criterion value

Step 4. Solution of the adjoint set of differential equations

Step 5. If the condition of convergence is satisfied, we will finish the calculating process. On the other hand, we will determine the gain of control and continue with the step 2.

If the process of successive approximations converges, we continue until the condition of $\boldsymbol{u}(t)$, $\boldsymbol{x}(t)$ defined according to the accuracy is satisfied. Thus, we will obtain the solution which satisfies the condition of convergence, it satisfies the maximum principle.

4 Solving the Optimal Control problem

The task is to determine $\boldsymbol{u}(t)$ and $\boldsymbol{x}(t)$ so that the cost function

$$J(\boldsymbol{u}(t), \boldsymbol{x}(t)) = \frac{1}{2}\int_{t_0}^{t_1} \left(\boldsymbol{x}_1^2(t) + \boldsymbol{x}_2^2(t) + 2\left(\boldsymbol{u}_1^2(t) + \boldsymbol{u}_2^2(t)\right)\right) dt$$

under the conditions and constraints fulfillment of the following form:

$$\frac{d\ x_1(t)}{dt} = -x_2(t) - u_1(t) - u_2(t)$$

$$\frac{d\ x_2(t)}{dt} = x_1(t) - u_1(t) + u_2(t)$$

$$|\boldsymbol{u}(t)| \le 1 ; \quad t \in \langle 1.57, 6.28 \rangle ;$$

$$\boldsymbol{x}^T(0) = (x_1(0), x_2(0)) = (1.50, 1.78),$$

obtains the minimum value.

The solution of the presented task, which is the mathematical model of the process optimum control, will use the programming system MS EXCEL – SOLVER, [4, 5, 6].

5 Conclusion

The article presents a mathematical formulation of the problem of the process or system optimal control. The Pontryagin's principle of maximum and the approximation method applied to determine the components of the control vector $\boldsymbol{u}(t)$ were used to solve the defined problem. The article also presents the algorithm of the successive approximations method as well as the ways (quickening) of the iteration process convergence improvement. The successive approximations method and its modifications can be used to solve another group of problems which terminal conditions for the control process and also corresponding cost function are defined. Finally, the article presents the solution of the problem of the process optimal control which is expressed by the mathematical model, i.e. by a system of differential equations and the optimal criterion is represented by the cost function. The program system MS EXCEL was used to solve the defined problem.

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Lektoroval: Ivan Taufer, prof. Ing., DrSc.

Kontaktní adresa: Department of Process Control and Computer Techniques, University of Pardubice, Nám. Čs. legií 565, 532 10 Pardzbice, Czech Republic, Tel.: +420466037123, e-mail: ivan.taufer@upce.cz

Kamil Hrubina, doc. Ing. Mgr., PhD.
Katedra matematiky, informatiky a kybernetiky
Fakulta výrobných technológií TU v Košiciach so sídlom v Prešove
Bayerova 1, 080 01 Prešov, SR, tel. +421 51 7723 931, fax +421 51 7733 453,
e-mail: hrubina.kamil@fvt.sk

Anna Jadlovská, doc. Ing., PhD.
Katedra kybernetiky a umelej inteligencie
Fakulta elektrotechniky a informatiky TU
Letná 9, 041 00 Košice, SR
e-mail: anna.jadlovska@tuke.sk