CYLINDRIC COORDINATES OTHERWISE

LIŠKA Vladimír – VRÁBEĽ Róbert, SR

Abstract

Substitution method to multidimensional integrals is a powerful tool for calculating them, but its routine application can lead to lengthy calculations. This fact presents the example of three-dimensional integral, where the area of integration is cone with the axis of rotation o_x.

Key words: three-dimensional integral, Fubini's theorem, cylindrical coordinates.

1 Introduction

Substitution (mapping) to the cylindrical coordinates in the standard form referred to in the following format.

$$\Sigma: \quad \begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ \mathrm{z} &= \mathrm{u}. \end{aligned} \tag{1}$$

Any point $X_0 = [x_0, y_0, z_0]$ of space E_3 can be expressed by the new coordinates ρ_0, ϕ_0 and u_0. Thus $X_0 = [\rho_0, \phi_0, u_0]$, where the of coordinates ρ_0, ϕ_0 and u_0 is meaning obvious from Figure 1.1, where ϕ_0 is the angle between o_x^+ and segment, which join the starting of coordinate system with orthogonal projections of points X_0 to the plane Π_{xy}, the point $X_0^{'}$, ρ_0 is the length of this segment and u_0 is z coordinate of point X_0. [1], [2], [3].

It is easy to verify that the mapping meets the substitution method into three-dimensional integrals and therefore in general, for an elementary area G and functions f the continuous on G holds

$$\iiint_G f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{G^*} f(\rho\cos\phi, \rho\sin\phi, u)\rho\mathrm{d}\rho d\phi du\,, \tag{2}$$

where the closed elemental area G^* is a solid G expressed in cylindrical coordinates (i.e. in the variables ρ, ϕ, u or otherwise $\Sigma(G^*) = G$). [1] [2], [3]

The following example shows the use of substitution in the cylindrical coordinates (1).

Example 1: Calculate

$$\iiint_G x\mathrm{d}xdydz\,,$$

where G is bounded by surfaces $x = \sqrt{y^2 + z^2}$ and $x = 4$.

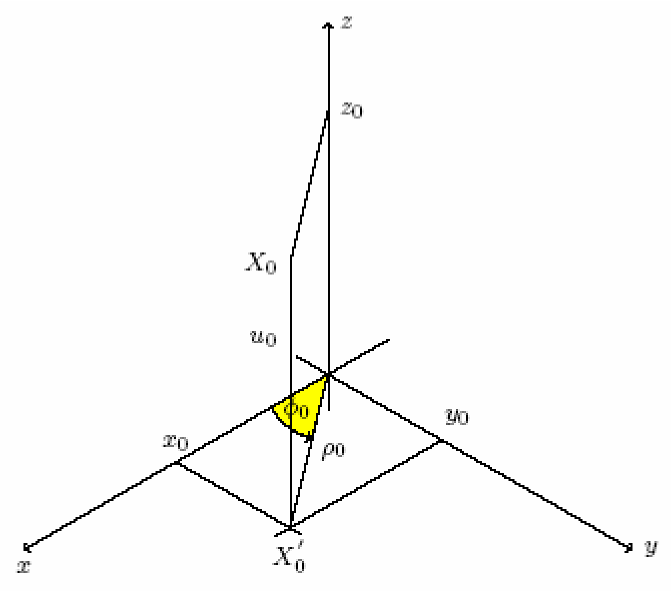

Figure 1.1: *Definition of cylindrical coordinates.*

Solution: Solid G is part of a cone with the axis of symmetry o_x (Figure 1.2). Orthogonal projection of the solid G in a plane Π_{xy} is shown on Figure 1.3. The area G^* may be described as elementary area of the type $EO_{[\phi,\rho,u]}$.

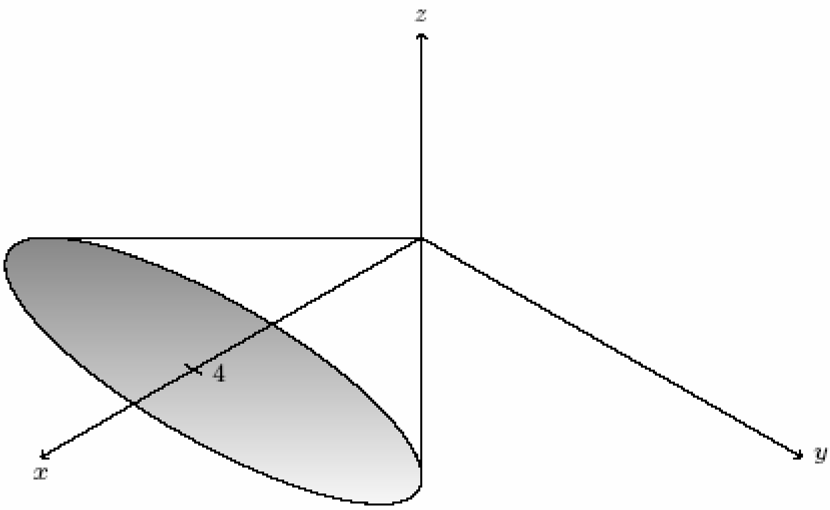

Figure 1.2: *Solid bounded by surfaces* $x = \sqrt{y^2 + z^2}$ *and* $x = 4$.

$$G^*: \quad \begin{aligned} -\frac{\pi}{4} \le \phi \le \frac{\pi}{4} \\ 0 \le \rho \le \frac{4}{\cos\phi} \\ -\rho\sqrt{\cos 2\phi} \le u \le \rho\sqrt{\cos 2\phi}. \end{aligned}$$

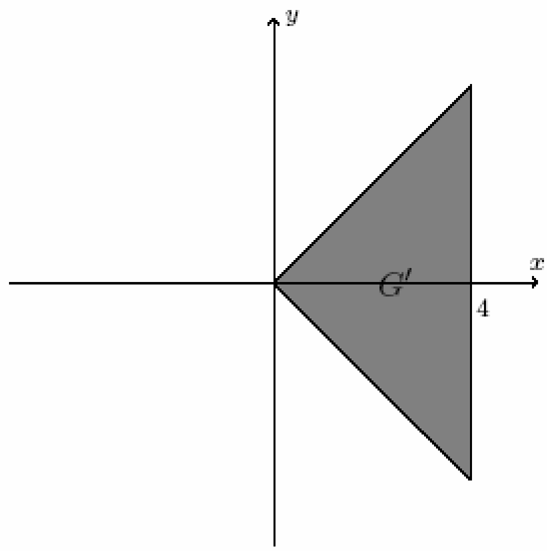

Figure 1.3: *Orthogonal projection G' of the solid G to the plane Π_{xy}.*

Now we use the substitution theorem to three-dimensional integrals and subsequently Fubini's theorem (the formulations of these theorems can be found in [3]). We obtain

$$\iiint_G x\mathrm{d}x dy dz = \iiint_{G^*} \rho\cos\phi\rho\mathrm{d}\rho d\phi du = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\frac{4}{\cos\phi}} \int_{-\rho\sqrt{\cos\phi}}^{\rho\sqrt{\cos\phi}} \rho^2\cos\phi\mathrm{d}u d\rho d\phi =$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\frac{4}{\cos\phi}} \rho^3\cos\phi\sqrt{\cos 2\phi} d\rho d\phi = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{\rho^4}{4}\right]_0^{\frac{4}{\cos\phi}} \cos\phi\sqrt{\cos 2\phi}\mathrm{d}\rho d\phi =$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{16.16}{\cos^3\phi}\sqrt{\cos 2\phi}\mathrm{d}\phi = 8.16\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sqrt{\cos 2\phi}}{\cos^3\phi}\mathrm{d}\phi = 16.16\int_0^{\frac{\pi}{4}} \frac{\sqrt{\cos 2\phi}}{\cos^3\phi}\mathrm{d}\phi =$$

$$= 16^2\int_0^{\frac{\pi}{4}} \frac{\sqrt{\cos^2\phi - \sin^2\phi}}{\cos^3\phi}\mathrm{d}\phi = 16^2\int_0^{\frac{\pi}{4}} \frac{\sqrt{1-\mathrm{tg}^2\phi}}{\cos^2\phi}\mathrm{d}\phi = \begin{vmatrix} t = \mathrm{tg}\phi \\ \mathrm{d}t = \dfrac{\mathrm{d}\phi}{\cos^2\phi} \end{vmatrix} =$$

$$= 16^2\int_0^1 \sqrt{1-t^2}\,\mathrm{d}t = \begin{vmatrix} t = \sin\omega \\ \mathrm{d}t = \cos\omega\mathrm{d}\omega \end{vmatrix} = 16^2\int_0^{\frac{\pi}{2}} \cos^2\omega\mathrm{d}\omega = 16^2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\omega}{2}\mathrm{d}\omega =$$

$$= \frac{16^2}{2}\left[\omega + \frac{\sin 2\omega}{2}\right]_0^{\frac{\pi}{2}} = \frac{16^2}{2}.\frac{\pi}{2} = 64\pi.$$

2 Alternatively cylindrical coordinates

Calculation of the example 1 is quite lengthy. Counting is simpler by using a modified substitution to the cylindrical coordinates

$$\Sigma^* : \begin{array}{l} x = u \\ y = \rho\cos\phi \\ z = \rho\sin\phi. \end{array} \qquad (3)$$

On the set $[\rho,\phi,u] \in (0,\infty)\times\langle 0,2\pi)\times R$, the mapping is regular and one – one with Jacobian

$$J_{\Sigma^*}(\rho,\phi,u) = \begin{vmatrix} 0 & 0 & 1 \\ \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \end{vmatrix} = \rho \neq 0. \qquad (4)$$

Then

$$\iiint_G f(x,y,z)\mathrm{d}xdydz = \iiint_{G^{**}} f(u,\rho\cos\phi,\sin\phi)\rho\mathrm{d}\rho d\phi du. \qquad (5)$$

<u>Solution Example 1, by using alternative cylindrical coordinates:</u>

$$\iiint_G x\mathrm{d}xdydz = \iiint_{G^{**}} u\rho\mathrm{d}\rho d\varphi du.$$

The orthogonal projection of the solid G in the plane Π_{yz}, determining the values of ρ and of the ϕ area G^{**}, is shown in Figure 2.1. The area described as the fundamental field type

$$G^{**}: \begin{array}{l} 0 \le \phi \le 2\pi \\ 0 \le \rho \le 4 \\ 0 \le u \le 4. \end{array}$$

The conditions for mapping Σ^* in the substitution theorem to the multidimensional integrals (Σ^* that is regular and one-one) may not be fulfilled at a set of measure zero, for example at the border of G^{**} (see [3], for example). Then

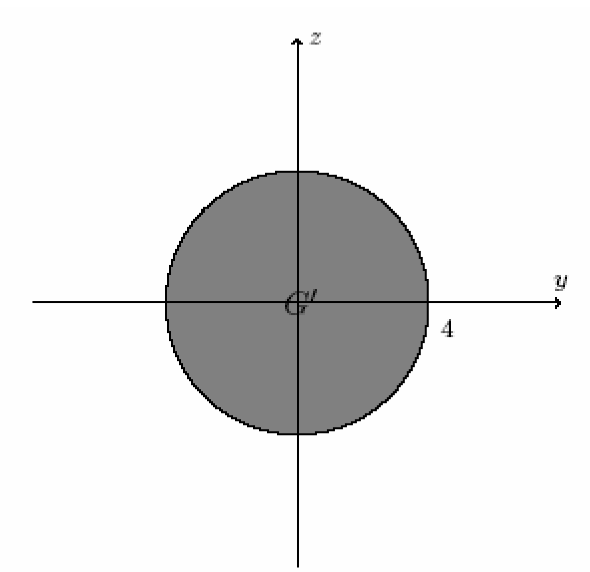

***Figure 2.1:** Orthogonal projection G' of the solid G to the plane.*

$$\iiint_G x\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{G^{**}} u\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}u =$$

$$\int_0^{2\pi}\int_0^4\int_\rho^4 u\rho\,\mathrm{d}u\mathrm{d}\rho\mathrm{d}\phi = \int_0^{2\pi}\int_0^4 \rho\left[\frac{u^2}{2}\right]_\rho^4 \mathrm{d}\rho\mathrm{d}\phi =$$

$$\frac{1}{2}\int_0^{2\pi}\int_0^4 \rho\left(16-\rho^2\right)\mathrm{d}\rho\mathrm{d}\phi = \frac{1}{2}\int_0^{2\pi}\left[8\rho^2 - \frac{\rho^4}{4}\right]_0^4 \mathrm{d}\phi =$$

$$\frac{1}{2}\int_0^{2\pi}(8.16 - \frac{4^4}{4})\mathrm{d}\phi = 32\left[\phi\right]_0^{2\pi} = 64\pi.$$

3 Conclusion

From the equality of the above results it follows that calculating example 1 using the transformation to cylindrical coordinates is correct for both methods. We see that there is quite significant difference in difficulty of solutions. To find a simply solution we have to choose the transformation depending on the particular example, what must be noted also in the teaching practice itself.

4 Literature

1. BARTSCH, H.J. *Matematické vzorce.* 1. vyd. Praha: SNTL – Nakladatelství technické literatury, n. p., 1983. 832 p.
2. HALABRÍN, M. – ČERVEŇANSKÝ, J. – PALUMBÍNY, O. *Matematika II.* 1. vyd. Bratislava: Vydavateľstvo Slovenskej technickej univerzity, 2005. 213 p. ISBN 80-227-2275-8.
3. KLUVÁNEK, I. – MIŠÍK, L. – ŠVEC, M. *Matematika II.* 1. vyd. Bratislava: Slovenské vydavateľstvo technickej literatúry, n. p., 1961. 856 p.

Acknowledgements: This paper has been supported by VEGA grant No. 1/0068/08

Lecturer: Jaroslava Trubenová, RNDr. PhD.

Contact address:

Vladimír Liška, Mgr.,
Ústav aplikovanej informatiky, automatizácie a matematiky,
Materiálovotechnologická fakulta STU
Hajdóczyho 1, 917 24 Trnava, SR, tel. 00421 918 646 021, fax 00421 335 447 736, e-mail vladimir.liska@stuba.sk

Róbert Vrábeľ, Mgr. PhD.,
Ústav aplikovanej informatiky, automatizácie a matematiky,
Materiálovotechnologická fakulta STU
Hajdóczyho 1, 917 24 Trnava, SR, tel. 00421 918 646 021, fax 00421 335 447 736, e-mail robert.vrabel@stuba.sk