SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS IN EDUCATION OF AUTOMATION

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Abstract

In the area of operations research, linear differential equations with constant coefficients very frequently occur. A standart way to solve such equations is using the method of variation of constants or Laplace transform. For some of linear differential equations with constant coefficients one can use both methods, while for the other is possible to use only one of them. In particular, if the right-hand side of the equation is a discontinuous function, it is not possible solve it by the method of variation of constants. On the other hand, if boundary conditions are of a special type, it is not possible to use Laplace transformation to solve the differential equation. In this contribution we show solutions in both cases – we show that the students of automation have to know the both ways of solving.

Keywords: Linear differential equation, Laplace transformation, variation of constants.

1 Introduction

Let we have to solve second order linear differential equation of the form

$$y''(t)+a_1y'(t)+a_2y(t)=g(t),\ \ a_1,a_2\in\Re. \tag{1}$$

If the right-hand side of the equation is continuous and the the problem is a boundary value problem, that is, one of the conditions $y(t_0)=a, y(t_1)=b$ or $y(t_0)=a, y'(t_1)=b$ or $y'(t_0)=a, y'(t_1)=b$ holds, one can solve the equation by the method of variation of constants. In this case (because of boundary conditions) it is not possible to solve it with Laplace transormation. On the other hand, if the function $g(t)$ is discontinuous, it is not possible use the method of variation of constants. If, in addition, the problem is an initial value problem, that is $y(t_0)=a, y'(t_0)=b,$ one can solve the equation with the help of Laplace transformation. We will show examples of solving the differential equation in both cases.

2 Boundary value problem, right-hand side is continuous

Problem: Solve differential equation $y''+9\mathrm{y}=\mathrm{t}^2+t+1,$ with conditions

$$y(0)=\frac{88}{81},\ \ y'(\pi)=\frac{2}{9}\pi.$$

Solution: This is a boundary value problem for an inhomogeneous linear differential equation of second order with constant coefficients, where the right-hand side of the equation is a continuous function. So we can not solve it with Laplace transform. Instead, we will use the method of undetermined coefficients. At first we solve the equation $y''+4\mathrm{y}=0$. The characteristic equation of that is $\lambda^2+4=0,$ so the characteristic roots are $\lambda_1=3i, \lambda_2=-3i.$ The two linearly independent solutions of the homogeneous differential equation are $y_1=\cos 3t$ and $\mathrm{y}_2=\sin 3t$ and the general solution of the homogeneous differential equation is $y_h=c_1y_1+c_2y_2=c_1\cos 3t+c_2\sin 3t.$

The function $g(t)$ on the right-hand side of the original equation is of a special form $g(t)=P_n(t)\cdot e^{t\alpha},$ where $n=2,$ $P_2(t)=t^2+t+1,$ $\alpha=0$ and α is not a root of the characteristic equation. Using the method of undetermined coefficients, the particular solution of the inhomogeneuous equation will be $y_p = at^2 + bt + c$ with undetermined coefficients a, b, c. When we substitute $y_p = at^2 + bt + c$ and $y_p'' = 2a$ into the original differential equation we obtain $a = \frac{1}{9}, b = \frac{1}{9}, c = \frac{7}{81}$. The general solution of the inhomogenous equation is the sum of the general solution of the homogeneous differential equation and the particular solution. So $y_g = y_h + y_p = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{9}t^2 + \frac{1}{9}t + \frac{7}{81}$, where c_1 and c_2 are an arbitrary real numbers. Substituting $y(0) = \frac{88}{81}$ and $y'(\pi) = \frac{2}{9}\pi$ onto $y_g(t)$ and $y_g' = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{2}t$ we obtain $c_1 = 1$ and $c_2 = \frac{1}{27}$. The solution of the boundary value problem on the interval $\langle 0; \pi \rangle$ is the function $y(t) = \cos 3t + \frac{1}{27}\sin 3t + \frac{1}{9}t^2 + \frac{1}{9}t + \frac{1}{8}$.

3 Initial value problem, right-hand side is discontinuous

Problem 2: Solve differential equation $y'' - 5y' + 6y = g(t)$, where the function $g(t)$ is given by: $g(t)\begin{cases} 0, \text{for } t \in (-\infty;0) \bigcup (1;\infty) \\ 1, \text{for } t \in \langle 0;1 \rangle \end{cases}$, with conditions $y(0) = 0, y'(0) = 2$.

Solution: The function $g(t)$ on the right-hand side of the equation is discontinuous on the interval $(0;\infty)$ so we can not use the method of variations of constants. Because the problem is an initial value problem (conditions are $y(0) = 0$ and $y'(0) = 2$) we can solve it using the Laplace transform. The images of y'', y', y and $g(t)$ are: $\mathcal{L}[y(t)] = Y(p)$, $\mathcal{L}[y'(t)] = p\mathcal{L}[y(t)] - y(0) = pY(p)$, $\mathcal{L}[y''] = p^2\mathcal{L}[y(t)] - py(0) - y'(0) = p^2Y(p) - 2$ and $\mathcal{L}[g(t)] = \int_0^\infty g(t)e^{-pt}dt = \int_0^1 1 \cdot e^{-pt}dt + \int^\infty 0 \cdot e^{-pt}dt = \int_0^1 e^{-pt}dt = -\frac{1}{p}\left[e^{-pt}\right]_0^1 = \frac{1}{p}\left(1 - e^{-p}\right)$. From the previous it follows that the image of the differential equation is $\left(p^2Y(p) - 2\right) - 3pY(p) + 2Y(p) = \frac{1}{p}\left(1 - e^{-p}\right)$. So the image of the solution is

$$Y(p) = \frac{2p+1}{p(p-1)(p-2)} - \frac{e^{-p}}{p(p-1)(p-2)}$$

which is (after partial fraction decomposition)

$$Y(p) = \frac{1}{2}\cdot\frac{1}{p} - 3\cdot\frac{1}{p-1} + \frac{5}{2}\cdot\frac{1}{p-2} - \frac{1}{2}\cdot\frac{e^{-p}}{p} + \frac{e^{-p}}{p-1} - \frac{1}{2}\cdot\frac{e^{-p}}{p-2}.$$

The original of $Y(p)$ is the function $y(t)$ - the solution of the differential equation is the original of. We have $y(t) = \mathcal{L}^{-1}[Y(p)] = \frac{1}{2} - 3e^t + \frac{5}{2}e^{2t} - \frac{1}{2}\eta(t-1) + e^{t-1}\eta(t-1) - \frac{1}{2}e^{2(t-1)}\eta(t-1)$ or equivalently

$$y(t) = \begin{cases} \frac{1}{2} - 3e^t + \frac{5}{2}e^{2t}, \text{for } t \in (-\infty;1) \\ \left(\frac{1}{e} - 2\right)e^t + \left(\frac{3}{2} - \frac{1}{2e^2}\right)e^{2t}, \text{for } t \in \langle 1;\infty) \end{cases}.$$

4 Conclusion

One of the main objects in the area of operations research are dynamical systems. These can be described by differential equations of various orders. Very often the differential equation is a linear differential equation of $n-$ th order with constant coefficients. For example, an RLC circuit can be described by differential equation $LCy''(t)+RCy'(t)+y(t)=g(t)$, where R - resistance, L - inductance and C - capacitance are positive real constants, voltage $g(t)$ is an input signal and $y(t)$ is the output signal. Trying to find the output signal (or, more generally, reaction of the system on the given input signal), students of automatization have to solve such differential equations. Because the right-hand side of the differential equation (the input signal) can be continuous as well as discontinuous and the imposed conditions can be of various types (boundary value problem or initial value problem), the students have to know to use both methods of solving – the method of variation of constants and Laplace transform. In this talk we showed how the solutions look like in both cases.

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6 References

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