LJAPUNOV STABILITY THEORY OF LINEAR AND NONLINEAR SYSTEMS AND TRANSFORMATION

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Abstract

The paper deals with the investigation of nonlinear systems stability, a characteristic exponent and asymptotic stability. It also deals with the Lyapunov transformation to carry out a linear system whose matrix elements are functions of a system with a constant matrix.

Key words: cybernetics, nonlinear systems stability, the Lyapunov transformation.

1 Introduction

There are several definitions of a nonlinear system stability. Many of them have a limited utilization and were defined for specific cases. In general, the processes going on in linear and nonlinear systems can be expressed by a mathematical model, which actually is a system of differential equations. Lyapunov stability theory enables to investigate the system stability without the necessity of solving either differential equations of the given order or a system of differential equations. A. Lyapunov proposed two methods in order to investigate the stability. Lyapunov first method enables to consider the nonlinear system stability according to an approximate linear model, (local stability). Lyapunov second method enables to consider the stability or the asymptotic stability in a certain area Ω, in general with the linear or nonlinear system, (of both excited and unexcited system). When solving the stability problem, the success of the method lies with the investigator'stability to find a suitable function (the so called Lyapunov function) as well as to determine its definiteness, [1, 2, 3].

This paper will deal with the investigation of nonlinear systems stability described by a vector differential equation, a characteristic exponent and an asymptotic stability. It will also deal with the Lyapunov transformation as well as the stability of the systems with variable coefficients of the system of differential equations.

2 System stability and characteristic exponent

We will consider a homogeneous linear vector differential equation (or a homogeneous linear system of differential equations) in the form

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}(t).\boldsymbol{x}(t) \tag{1}$$

where matrix individual elements

$$\boldsymbol{A}(t)=(a_{ij}(t)) \tag{2}$$

are continuous functions in the interval $(a, +\infty)$.

Theorem 1. *A* liner system described by the equation (1) is stable in the sense of Lyapunov in the interval $< t_0, +\infty >$, if all the solutions to the equation (1) are bounded functions in the interval $< t_0, +\infty >$.

Theorem 2 (R. Bellman). Let all the solutions to the vector differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}.\boldsymbol{x}(t) \tag{3}$$

with a constant matrix of (*n , n*) type be stable in the sense of Lyapunov, or let all the roots of the equation *det* ***A*** = 0 have negative real parts. Let ***B***(*t*) be a matrix, whose elements are variables of the

(*n, n*) type, where its elements are continuous functions in the interval $< t_0, +\infty >$. and let the integral be expressed in the form

$$\int_{t_0}^{\infty} \|\boldsymbol{B}(\tau)\| . d\tau < 0, \tag{4}$$

then all the solutions to the equation

$$\dot{\boldsymbol{y}}(t) = [\boldsymbol{A} + \boldsymbol{B}(\tau)].\boldsymbol{y}(t) \tag{5}$$

are stable in the interval $< t_0, +\infty >$ in the sense of Lyapunov.

2.1 Characteristic exponent

First, we will present stability conditions of linear systems with variable coefficients. The basic notion is that of a characteristic exponent of the function introduced by A.M. Lyapunov.

Definition 1. A characteristic exponent of a complex function *f*(*t*) of a real variable *t* is called a number.

$$\chi(f) = \lim_{t \to \infty} \sup (t)^{-1} . \ln |f(t)| \tag{6}$$

In order to explain this notion by which a growth velocity of a function is characterized, it is sufficient to realize the following fact. A module of the given function can be expressed in the form

whereas $|f(t)| = e^{\alpha(t).t}$, whereas $\alpha(t) = (t)^{-1}.\ln|f(t)|$
(7)

Theorem 3. (A. M. Ljapunov). If the matrix (2) in the equation (1) is norm - bounded (we can assume an arbitrary norm in the relation (11)).

$$\|\boldsymbol{A}(t)\| \le C < +\infty , \tag{8}$$

then every non-zero solution ***x***(*t*) has the infinite characteristic exponent.

Theorem 4. In the sense of Lyapunov, for the asymptotic stability of a linear homogeneous system described by the vector equation (1) it is sufficient that its maximum characteristic exponent is negative.

3 Lyapunov transformation

When investigating the stability of solutions to homogeneous linear systems (1) in some cases it is possible to find a linear transformation

$$\boldsymbol{y}(t) = \boldsymbol{L}(t).\boldsymbol{x}(t) \tag{9}$$

which will change a system (1) with the ***A***(*t*) matrix to the system

$$\frac{d\boldsymbol{y}(t)}{dt} = \boldsymbol{B}\boldsymbol{y}(t) \tag{10}$$

with a constant matrix. If during this transformation the characteristic exponents are not changed, it is possible to solve the problem of stability of the system (1) by means of known methods.

Definiton 2. The $\boldsymbol{L}(t)$ matrix, whose elements have continuous first derivatives in the interval $< t_0, +\infty)$, is called Lyapunov matrix, if

a) $\sup\limits_{t \in < t_0, +\infty)} \|\boldsymbol{L}(t)\|$ *and* $\sup\limits_{t \in < t_0, +\infty)} \left\|\frac{d\boldsymbol{L}(t)}{dt}\right\|$ are finite numbers b) $\|det\,\boldsymbol{L}(t)\| \geq k > 0 \quad \forall\, t \in < t_0, +\infty)$

The corresponding transformation (9) is called Lyapunov transformation.

4 Stability of systems with variable coefficients

Differential equations which are used to describe the systems with variable parameters, have time-varying coefficients; they will be denoted as $a_i(t)$ time functions. The stability of the systems with variable parameters can be secured only in a certain time interval. Beyond this interval, the system can be instable.

4.1 Basic relations

We will investigate the system with variable parameters described by the differential equation

$$a_n(t)x_2^{(n)}(t) + ... + a_1(t)x_2'(t) + a_0(t)x_2(t) = x_0(t) \qquad (11)$$

$x_2(t)$ is an output value, $x_0(t)$ is an input value. Our task is to find a relation between the input and the output values of the investigated system for such a case that the system is in an equilibrium until the moment when the input signal starts acting,. The solution is considered from the moment when the input signal is applied. For this moment it holds:

$$x_2^{(\nu)}(t)\,|_{t=0} = 0, \quad \nu = 1,2,....,(n-1) \qquad (12)$$

The solution to the equation (11) will be obtained by the variation of constants method, [3]. The considered solution is searched for in the form

$$x_2(t) = \varphi_1(t)\gamma_1(t) + \varphi_2(t)\gamma_2(t) + ... + \varphi_n(t)\gamma_n(t) \qquad (13)$$

where $\varphi_i(t)$ are linear independent particular solutions to the homogeneous equation, $\gamma_j(t)$ will be determined in such a way that after inserting the expression (13) into (11) we obtain the identity.

Thus we obtain the solution in the form:

$$x_2(t) = \int_0^t g(t,u)x_0(u)du \qquad (14)$$

In order to explain the physical substance of the $g\,(t,\,u)$ function , we will investigate the case in which at the moment $t = \xi$ for the system input there is introduced a signal in the form of the Dirac impulse, i. e.

$$x_0(u) = \delta(u - \xi), \quad 0 < \xi < u$$

If we apply the expression (14) and the known equality

$$\int_{-\infty}^{\infty} f(t).\delta(t-\xi)dt = f(\xi) \qquad \int_{0}^{t} g(t,u).\delta(t-\xi)du = g(t,\xi)$$

thus we obtain an impulse transition function of the system, which is described by the equation (13).

The impulse transition function is called the system response (which before the beginning of the signal acting was in an equilibrium) to the input signal in the form of the Dirac impulse. Considering a mathematical point of view, $g(t,\xi)$ is the solution to the differential equation

$$a_n(t)g^{(n)}(t,\xi) + ... + a_1(t)g'(t,\xi) + a_0(t)g(t,\xi) = \delta(t-\xi) \tag{15}$$

with the initial conditions

$$g^{(\nu)}(t,\xi)|_{t=\xi} = 0, \qquad \nu = 0,1,2,...,(n-1)$$

The impulse transition function can also be applied to a more general case for the systems with changeable parameters to solve the equation in the form:

$$\begin{aligned} &a_n(t)w^{(n)}(t,\xi) + ... + a_1(t)w'(t,\xi) + a_0(t)w(t,\xi) = \\ &= b_m(t)\delta_m^{(m)}(t-\xi) + ... + b_1(t)\delta_1'(t-\xi) + b_0(t)\delta_0(t-\xi) \end{aligned} \tag{16}$$

with the initial conditions

$$w^{(\nu)}(t,\xi)|_{t=\xi} = 0, \qquad \nu = 0,1,2,...,(n-1) \tag{17}$$

In this case, $w(t,\xi)$ represents an impulse transition function of the system with the changeable parameters of a general form. The $w(t,\xi)$ function is related to the $g(t,\xi)$ function according to the relation

$$w(t,\xi) = (-1)^m \frac{d^m}{d\xi^m}\left[g(t,\xi)b_m(\xi)\right] + ... + g(t,\xi)b_0(\xi) \tag{18}$$

5 Conclusion

In order to solve the problems of stability defined by a linear vector differential equation with the $A(t)$ changeable matrix, Bellman, Gronwald and Lyapunov lemmas and theorems were applied. This refers to the theorems utilizing the defined notion of a characteristic exponent, a matrix spectrum, but especially the Lyapunov transformation.

The essence of the presented methods of solution applied to the problems of asymptotic stability of the system with a time varying matrix lies in the application to the problem of the system stability with a constant matrix. The mentioned possibility of the problem solving is proved by means of the Lyapunov transformation and Levinson theorem. Another contribution lies in the solution of the problems of stability of the systems with changeable parameters which are described by a system of differential equations with time-varying coefficients.

The paper was designed based on the grant support *VEGA No. 1/0345/08: Modeling and Simulation of Mechatronic Systems for Mechanical Engineering,,* as well as by the *Agency for the EU Structural Funds of the Ministry of Education of Slovak Republic under the project : Centre of Information and Communication Technologies for Knowledge Systems (project number 26220120020).*

6 Literature

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