NOTE ON POLAR COORDINATES

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Abstract

This article deal with a appropriate use of polar coordinates. We show by an example that routine transformation of Cartesian coordinates to polar coordinates in a two-dimensional integral can lead to lengthy calculation. Appropriate use of polar coordinates on the same example will substantially simplify the calculations.

Key words: two-dimensional integral, Fubini's theorem, polar coordinates.

1 Introduction

Most common transformation (mapping) $A = \Phi(A^*)$ Cartesian coordinates to polar coordinates, which each point $A^* = [\rho_0, \varphi_0]$ of the set E_2 assigns to point $A = [a1, a2]$ of the space E_2 is given by equations

$$\Phi : \begin{matrix} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{matrix} . \qquad (1)$$

Meaning of polar coordinates is obvious from Figure 1, where ρ_0 is the length of segment, which join the origin of coordinate system with point A and φ_0 is the angle between o_x^+ and this segment.[1],[2],[3]

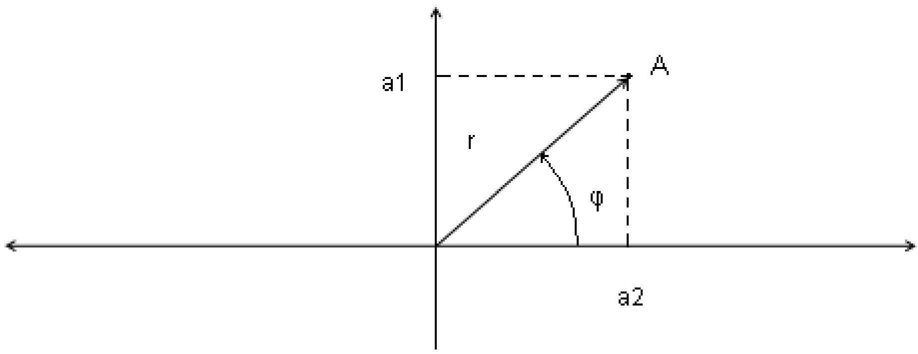

Figure 1: Polar coordinates

This is one-to-one mapping on the set $\Omega^* \subset \{[\rho, \varphi] \in E_2 : \rho > 0, 0 < \varphi < 2\pi\}$, i.e. for every two different points A_1^*, $A_2^* \in G^*$ exist the distinct images A_1, A_2. For Jacoby determinant hold

$$J_\Phi(\rho, \varphi) = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho . \qquad (2)$$

If $G = \Phi(G^*), G^* \subset \Omega^*$ is a closed elementary area (i.e. measurable set) and the function $f(x, y)$ is integrable over area G, then hold

$$\iint_G f(x,y)dxdy = \iint_{G^*} f(\rho\cos\varphi, \rho\sin\varphi) J_\Phi(\rho,\varphi) d\rho d\varphi . \tag{3}$$

The following illustrative example is solved by most common using the polar coordinates.

2 Example

Solve:

$$\iint_G dxdy$$

where G is area bounded by curve $(x-x_0)^2 + y^2 = r^2$.

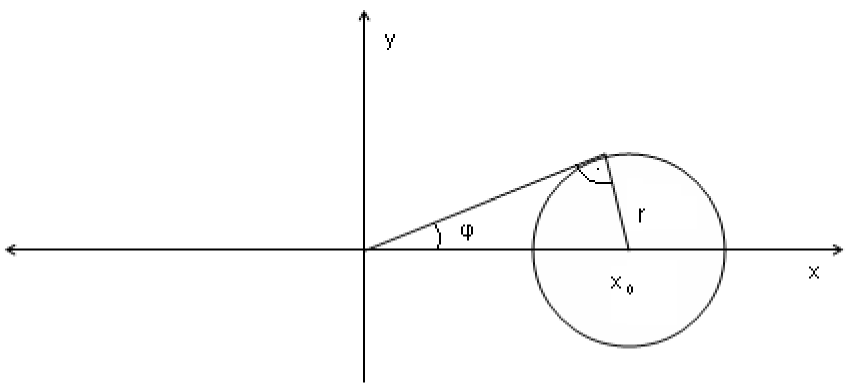

Figure 2: Area bounded by curve $(x-x_0)^2 + y^2 = r^2$

Solution (first approach): The area G is a part of Euclidean plane enclosed by circle with center in the point $[x_0, 0]$ and radius r, we can see that from figure 2. This area can be described in the polar coordinates, i.e. G^* is given by the inequalities

$$x_0\cos\varphi - \sqrt{r^2 - x_0^2\sin^2\varphi} \le \rho \le x_0\cos\varphi + \sqrt{r^2 - x_0^2\sin^2\varphi}$$

$$-\arcsin\frac{r}{x_0} \le \varphi \le \arcsin\frac{r}{x_0}.$$

Now we integrate by the substitution theorem to two-dimensional integrals and subsequently Fubini's theorem. We obtain

$$\iint_G dxdy = \iint_{G^*} \rho d\rho d\varphi = \int_{-\arcsin\frac{r}{x_0}}^{\arcsin\frac{r}{x_0}} d\varphi \int_{x_0\cos\varphi-\sqrt{r^2-x_0^2\sin^2\varphi}}^{x_0\cos\varphi+\sqrt{r^2-x_0^2\sin^2\varphi}} \rho d\rho = \int_{-\arcsin\frac{r}{x_0}}^{\arcsin\frac{r}{x_0}} d\varphi \left[\frac{\rho^2}{2}\right]_{x_0\cos\varphi-\sqrt{r^2-x_0^2\sin^2\varphi}}^{x_0\cos\varphi+\sqrt{r^2-x_0^2\sin^2\varphi}} =$$

$$\int_{-\arcsin\frac{r}{x_0}}^{\arcsin\frac{r}{x_0}} 2x_0\cos\varphi\sqrt{r^2-x_0^2\sin^2\varphi}\,d\varphi = \left| \begin{matrix} t=\frac{x_0}{r}\sin\varphi & t_1=1 \\ \sin\varphi = t\frac{r}{x_0} & \\ dt=\frac{x_0}{r}\cos\varphi d\varphi & t_2=-1 \end{matrix} \right| = 2r^2\int_{-1}^{1}\sqrt{1-t^2}\,dt =$$

$$\left| \begin{matrix} t=\sin s & s_1=\frac{\pi}{2} \\ dt=\cos s\,ds & s_2=-\frac{\pi}{2} \end{matrix} \right| = 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 s\,ds = r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+\cos 2s\,ds = r^2\left[s+\frac{\sin 2s}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi r^2.$$

3 Other transformation to polar coordinates

Now we will solve the same example by using other transformation to the polar coordinates of the form

$$\tilde{\Phi}: \begin{matrix} x=\rho\cos\varphi \\ y=x_0+\rho\sin\varphi \end{matrix}. \quad (4)$$

This is one-to-one mapping on the set $\Omega^* \in (0,\infty)\times\langle 0,2\pi)$. For Jacoby determinant we have

$$J_{\Phi^*}(\rho,\varphi) = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho \neq 0. \quad (5)$$

Solution with alternative transformation (4) (second approach)**:** In new polar coordinates $G=\tilde{\Phi}\left(\tilde{G}^*\right)$ by the inequalities where $\tilde{G}^*$ is given

$$\tilde{G}^*: \begin{matrix} 0\le\rho\le 2\pi \\ 0\le\varphi\le r. \end{matrix}$$

We obtain

$$\iint_G dxdy = \iint_{\tilde{G}^*} \rho d\rho d\varphi = \int_0^{2\pi} d\varphi \int_0^r \rho d\rho = \int_0^{2\pi} d\varphi \left[\frac{\rho^2}{2}\right]_0^r = \int_0^{2\pi}\frac{r^2}{2}d\varphi = \left[\frac{r^2}{2}\varphi\right]_0^{2\pi} = \pi r^2.$$

4 Conclusion

We know that the circle surface area is πr^2, which is the result of our example. For both used transformation to polar coordinates we got the same correct results of our example. However, differences is in the difficulty and length calculation. The second method of calculation is considerably simpler. To find a simply solution we have to choose the transformation depending on the particular example. When explaining the transformation of Cartesian coordinates to polar coordinates in the two-dimensional integrals, is necessary this fact show to the students.

5 Literature

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